13 Fresnel Diffraction

In this section we will look at the Fresnel diffraction for both circular apertures and rectangular apertures. To help our physical understanding we will begin our discussion by describing Fresnel zones.

13.1 Fresnel Zones

In the study of Fresnel diffraction it is convenient to divide the aperture into regions called Fresnel zones. Figure 1 shows a point source, S, illuminating an aperture a distance $z_1$away. The observation point, P, is a distance to the right of the aperture. Let the line SP be normal to the plane containing the aperture. Then we can write

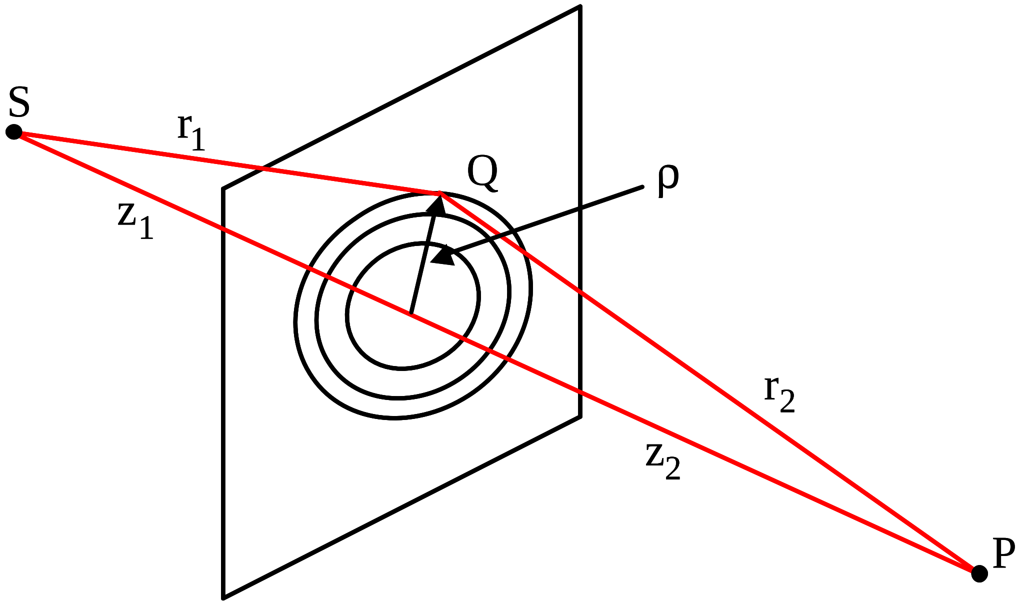


Fig. 1. Spherical wave illuminating aperture.

$$\begin{aligned}\text{SQP} &= r_1 + r_2 = \sqrt{z_1{}^2 + \rho^2} + \sqrt{z_2{}^2 + \rho^2} \\ &= z_1 + z_2 + \frac{1}{2}\rho^2\left(\frac{1}{z_1} + \frac{1}{z_2}\right) + \cdots\end{aligned}$$

The aperture can be divided into regions bounded by concentric circles ρ = constant defined such that $r_1 + r_2$ differ by $\lambda/2$ in going from one boundary to the next. These regions are called Fresnel zones or half-period zones. If $z_1$and z_2 are sufficiently large compared to the size of the aperture the higher order terms of the expansion can be neglected to yield the following result.

$$n\frac{\lambda}{2} = \frac{1}{2}\rho_n{}^2\left(\frac{1}{z_1} + \frac{1}{z_2}\right)$$

Solving for ρ_n, the radius of the nth Fresnel zone, yields

$$\rho_n = \sqrt{n \lambda L} \text{ or } \rho_1 = \sqrt{\lambda L},\ \rho_2 = \sqrt{2 \lambda L},\ \cdots, \text{ where} \qquad (1)$$

$$L = \frac{1}{\frac{1}{z_1} + \frac{1}{z_2}}$$

Figure 2 shows a drawing of Fresnel zones where every other zone is made dark. Note that in the center the zones are widely spaced and the spacing decreases as the radius increases. As shown above, the radius of the zones increase as the square root of integers.

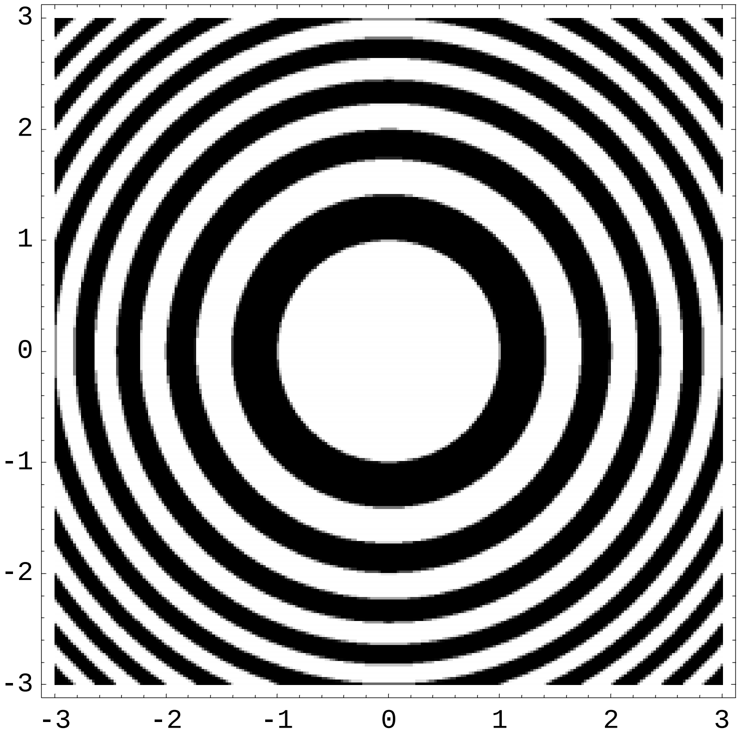


Fig. 2. Fresnel zones.

If ρ_n and $\rho_{n+1}$are inner and outer radii of the nth zone then the area of the nth zone is given by

$$\begin{aligned}\text{Area of nth Fresnel zone} &= \pi \rho_{n+1}{}^2 - \pi \rho_n{}^2 \\ &= \pi (n+1) \lambda L - \pi (n) \lambda L \qquad (2) \\ &= \pi \lambda L = \pi \rho_1{}^2, \text{ independent of n.}\end{aligned}$$

That is, the area of all zones are equal. If the higher order terms in the expansion for SRQ are maintained the area of the zones would slightly increase with increasing ρ. Generally, it is assumed that $z_1$and z_2 are sufficiently large compared to ρ that the higher order terms can be neglected and the area of all zones are equal.

13.2 Circular Aperture

On-Axis Irradiance

In discussing the Fresnel diffraction of a circular aperture it is convenient to first look at the on-axis irradiance. Consider a circular aperture of radius R, and let S and P lie on the normal through the center of the circular aperture as shown in Figure 3.

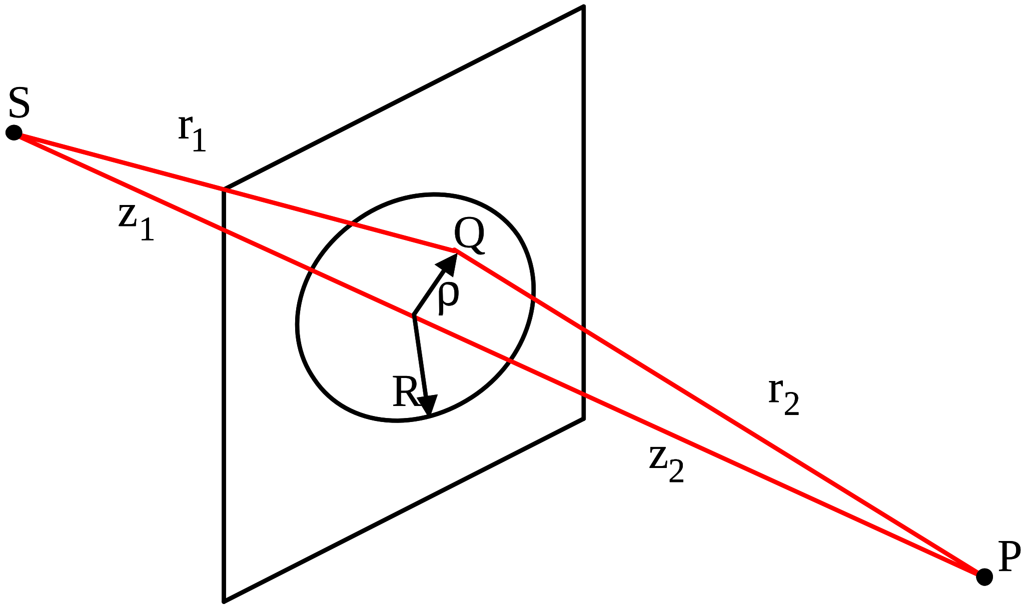


Fig. 3. Circular aperture of radius R.

If R is small compared with $z_1$and z_2 the usual approximations can be made in the Kirchhoff integral that the obliquity factor is equal to 1 and the $1/r_1\,r_2$ in the denominator can be replaced with $1/z_1\,z_2$. The cylindrical symmetry about the axis SP suggests using polar coordinates in the integration. Then

$$u_o[P] = \frac{A}{i\,\lambda\,z_1\,z_2}\int_0^R e^{i\,k\,(r_1+r_2)}\,2\,\pi\,\rho\,d\rho \tag{3}$$

Since $r_1^2 = z_1^2 + \rho^2$ and $r_2^2 = z_2^2 + \rho^2$,

$$\rho\,d\rho = r_1\,d r_1 = r_2\,d r_2$$

$$d(r_1 + r_2) = \left(\frac{1}{r_1} + \frac{1}{r_2}\right)\rho\,d\rho$$

Therefore,

$$\rho\,d\rho = \frac{r_1\,r_2}{r_1 + r_2}\,d(r_1 + r_2) \approx \frac{z_1\,z_2}{z_1 + z_2}\,d(r_1 + r_2)$$

$$u_o[P] = \frac{2\pi A}{i\lambda(z_1+z_2)}\int_{l[0]}^{l[R]} e^{ik(r_1+r_2)}\,d(r_1+r_2)$$

where $l[0] = z_1 + z_2$ and $l[R]$ is equal to $r_1 + r_2$ when $\rho = R$, that is when point Q is on the rim of the aperture.

It is convenient to introduce a variable q defined by

$$r_1 + r_2 = z_1 + z_2 + q\,\frac{\lambda}{2}$$

Thus q measures the difference between $r_1 + r_2$ and $z_1 + z_2$ in units of half-wavelengths and is equal to the number of Fresnel zones within the circular aperture. $u_o[p]$ can now be written as

$$u_o[P] = \frac{\pi\,u_{oo}[P]}{i}\int_0^{\zeta} e^{i\pi q}\,dq$$

where ζ=q[R], that is the value of q associated with a point on the rim of the aperture, and $u_{oo}[p]$ is the unobstructed value of the amplitude at point P.

$$u_{oo}[P] = \frac{A}{z_1+z_2}\,e^{ik(z_1+z_2)}$$

Solving the integral yields

$$u_o[P] = u_{oo}[P]\,(1 - e^{i\pi\zeta}) \qquad (4)$$

The on-axis irradiance is given by

$$i[P] = 2\,i_{oo}[P]\,(1 - \mathrm{Cos}[\pi\zeta]) = 4\,i_{oo}[p]\,\mathrm{Sin}\left[\frac{\zeta\pi}{2}\right]^2 \qquad (5)$$

As the radius of the aperture increases, the irradiance at point P oscillates between zero and 4 times the unobstructed irradiance. Thus, the irradiance produced by the first Fresnel zone is equal to 4 times the irradiance obtained with a clear aperture. As the equation shows, for odd numbers of Fresnel zones present

$$i[P] = 4\,i_{oo}[P] \qquad (6)$$

For even numbers of Fresnel zones present the Fresnel zones cancel one another and the on-axis irradiance is zero.

The relationship between ζ and R is found as follows: At the edge of the aperture ρ = R and

$$r_1 = (z_1^2 + R^2)^{1/2} \approx z_1 + \frac{R^2}{2\,z_1} \text{ and}$$

$$r_2 = (z_2^2 + R^2)^{1/2} \approx z_2 + \frac{R^2}{2\,z_2}$$

But

$$r_1 + r_2 = z_1 + z_2 + \zeta\,\frac{\lambda}{2}$$

Solving for ζ, the number of Fresnel zones present, yields

$$\zeta = \frac{R^2\,(z_1 + z_2)}{\lambda\, z_1\, z_2} \tag{7}$$

Off-Axis Irradiance Distribution

Now we want to calculate the irradiance of the Fresnel diffraction of a circular aperture at off-axis positions. Fig. 4 shows the schematic setup.

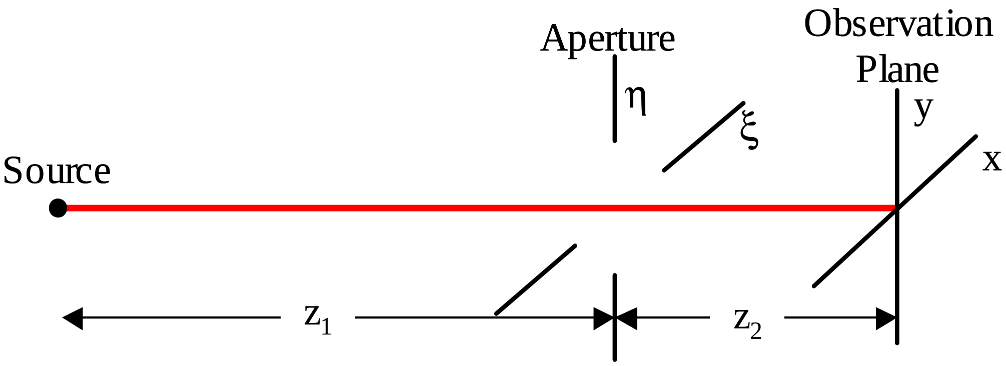


Fig. 4. Fresnel diffraction of circular aperture.

The general equation for the amplitude of the Fresnel diffraction pattern is given by

$$u[x, y] = \frac{e^{i\,k\,z_2}}{i\,\lambda\,z_2}\, e^{i\,\frac{\pi}{\lambda z_2}\,(x^2+y^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(u[\xi, \eta]\, e^{i\,\frac{\pi}{\lambda z_2}\,(\xi^2+\eta^2)}\right) e^{-i\,\frac{2\pi}{\lambda z_2}\,(x\,\xi + y\,\eta)}\, d\xi\, d\eta \tag{8}$$

One important fact to note about this equation is that we have the Fourier transform of $u[\xi, \eta]\, e^{i\,\frac{\pi}{\lambda z_2}\,(\xi^2+\eta^2)}$, where $u[\xi, \eta]$ is the amplitude of the wavefront illuminating the aperture which we will take as the amplitude of a spherical wave. For an on-axis spherical wave illuminating the aperture $u[\xi, \eta]$ can be written as $\frac{A}{z_1} e^{i\,k\,z_1}\, e^{i\,\frac{k}{2 z_1}\,(\xi^2+\eta^2)}$. If Pupil$[\xi, \eta]$ describes the aperture amplitude transmission function we can write

$$u[x, y] = A\,\frac{e^{i\,k\,(z_1+z_2)}}{i\,\lambda\,z_1\,z_2}\, e^{i\,\frac{\pi}{\lambda z_2}\,(x^2+y^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\text{Pupil}[\xi, \eta]\, e^{i\,\frac{\pi\,(z_1+z_2)}{\lambda\,z_1\,z_2}\,(\xi^2+\eta^2)}\right) e^{-i\,\frac{2\pi}{\lambda z_2}\,(x\,\xi + y\,\eta)}\, d\xi\, d\eta \tag{9}$$

Let $r_{\max}$ be the maximum value of $\sqrt{(\xi^2 + \eta^2)}$ and nFZones be the number of Fresnel zones in the aperture. Then

$$\text{nFZones} = \frac{(z_1 + z_2)}{\lambda\, z_1\, z_2}\, r_{\max}^2 \text{ and we can write}$$

$$u[x,\ y] = A\ \frac{e^{i\,k\,(z_1+z_2)}}{i\,\lambda\,z_1\,z_2}\ e^{i\,\frac{\pi}{\lambda z_2}\,(x^2+y^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\text{Pupil}[\xi,\ \eta]\ e^{i\,(\text{nFZones}\,\pi)\,\frac{(\xi^2+\eta^2)}{r_{\max}{}^2}}\right) e^{-i\,\frac{2\pi}{\lambda z_2}\,(x\,\xi+y\,\eta)}\ d\xi\, d\eta \qquad (10)$$

But $\frac{r_{\max}{}^2}{\lambda\,z_1\,z_2} = \frac{\text{nFZones}}{(z_1+z_2)}$, so

$$u[x,\ y] = A\ \frac{e^{i\,k\,(z_1+z_2)}}{i\,(z_1+z_2)}\ \frac{\text{nFZones}}{r_{\max}{}^2}\ e^{i\,\frac{\pi}{\lambda z_2}\,(x^2+y^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\text{Pupil}[\xi,\ \eta]\ e^{i\,(\text{nFZones}\,\pi)\,\frac{(\xi^2+\eta^2)}{r_{\max}{}^2}}\right) e^{-i\,\frac{2\pi}{\lambda z_2}\,(x\,\xi+y\,\eta)}\ d\xi\, d\eta \qquad (11)$$

$$u[x,\ y] = U_{oo}\ \frac{\text{nFZones}}{i\,r_{\max}{}^2}\ e^{i\,\frac{\pi}{\lambda z_2}\,(x^2+y^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\text{Pupil}[\xi,\ \eta]\ e^{i\,(\text{nFZones}\,\pi)\,\frac{(\xi^2+\eta^2)}{r_{\max}{}^2}}\right) e^{-i\,\frac{2\pi}{\lambda z_2}\,(x\,\xi+y\,\eta)}\ d\xi\, d\eta \qquad (12)$$

where $U_{oo} = A\ \frac{e^{i\,k\,(z_1+z_2)}}{(z_1+z_2)}$ is the unobstructed on-axis irradiance.

Fourier-Bessel transform (Hankel transform of zero order)

Since the aperture is circular we have the Fourier transform of a rotationally symmetric function which can be written as a Fourier-Bessel transform (Hankel transform of zero order). If the irradiance is normalized so the result is in units of the unobstructed irradiance, the irradiance as a function of the number of Fresnel zones in the aperture can be written as

$$\textbf{irradianceBessel}[\textbf{nFZones\_},\ \textbf{r\_}] := \textbf{Abs}\left[\textbf{nFZones}\int_0^1 \textbf{BesselJ}[0,\ \pi\,r\,\rho]\ e^{i\,(\text{nFZones}\,\pi)\,\rho^2}\ 2\,\pi\,\rho\, d\rho\right]^2$$

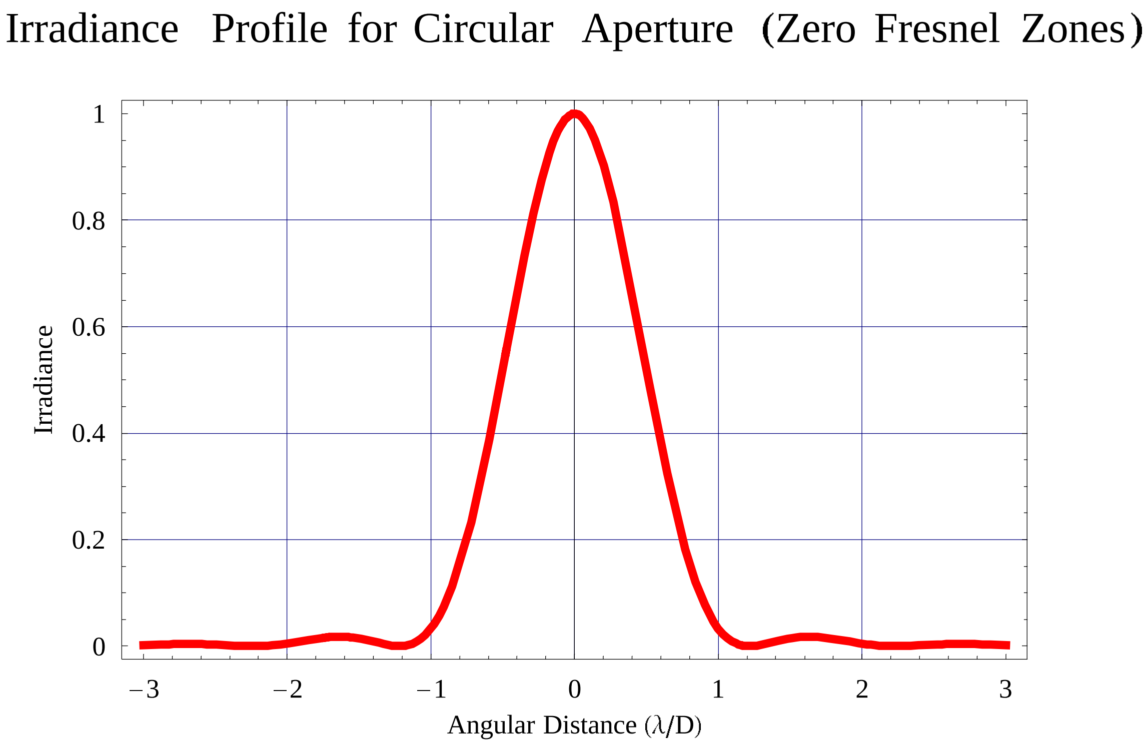
The pupil radius, ρ, is normalized to one at the edge of the aperture and the units of r are wavelengths/(pupil diameter). Sometimes the r in the above equation is replaced with 1.22 r, in which case r has the units of Airy disk radii.

If the irradiance is normalized so zero Fresnel zones present (Fraunhofer diffraction) give a maximum irradiance of 1, the irradiance as a function of the number of Fresnel zones in the aperture can be written as

$$\textbf{irradianceBesselUnity}[\textbf{nFZones\_},\ \textbf{r\_}] := \textbf{Abs}\left[\int_0^1 \textbf{BesselJ}[0,\ \pi\,r\,\rho]\ e^{i\,(\text{nFZones}\,\pi)\,\rho^2}\ 2\,\rho\, d\rho\right]^2$$

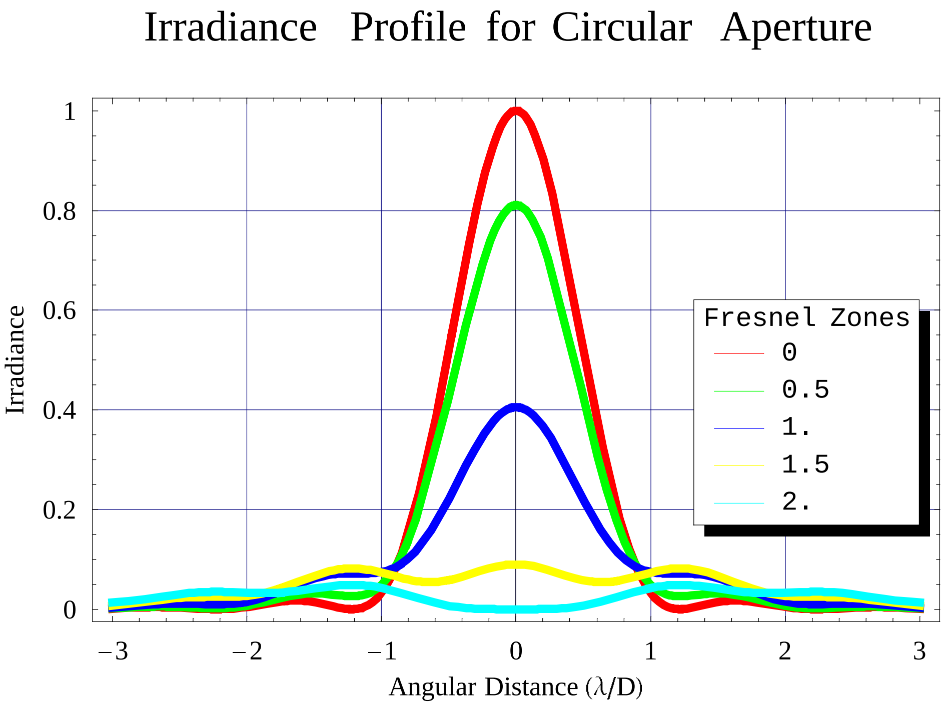
This expression is useful if we are looking at the irradiance distribution produced by a lens as a function of defocus.

Now we want to look at several examples of Fresnel diffraction of a circular aperture containing various numbers of Fresnel zones. For the first example we will look at the situation where there are zero Fresnel zones present, that is Fraunhofer diffraction.



The results looks like the normal $(2\,J_1(x)/x)^2$ that is obtained for Fraunhofer diffraction.

For the next example we will look at 0, 0.5, 1.0, 1.5, and 2 Fresnel zones in the aperture.



We showed above that if the size of the aperture is allowed to increase as the number of Fresnel zones present in the aperture increases such that the aperture area is proportional to the number of Fresnel zones, the on-axis irradiance varied from zero to 4 times the unobstructed irradiance. We now want to look at how the on-axis irradiance varies as a function of number of Fresnel zones present if the aperture size is held constant. This calculation will show how the Strehl Ratio (max irradiance of point source in presence of aberration divided by max irradiance in absence of aberration) for an optical system varies as the optical system is defocused.

If the size of the aperture is held constant as the number of Fresnel zones within the aperture is varied the on-axis irradiance as a function of the number of Fresnel zones within the aperture is given by

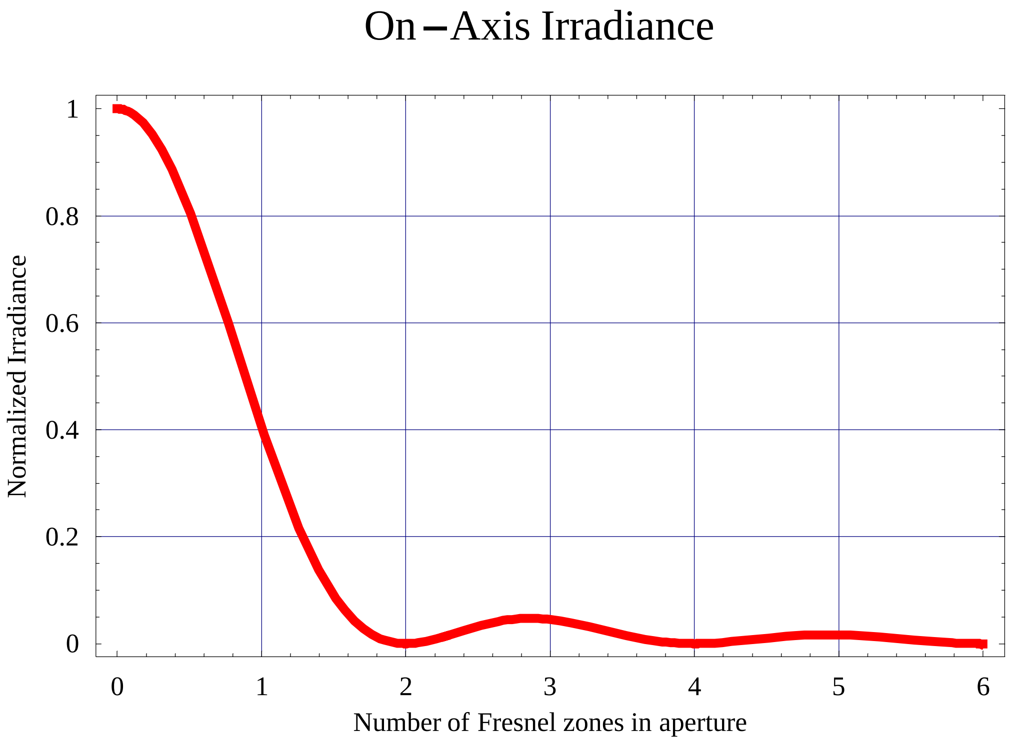
$$\text{onAxisIrradiance[nFZones\_]} := \text{Abs}\left[\int_0^1 e^{i\,(\text{nFZones}\,\pi)\,\rho^2}\, 2\,\rho\, d\rho\right]^2$$

The above equation is normalized so the on-axis irradiance for zero zones present is unity. Solving the above integral gives

$$\text{If nFZones} = 0, \qquad \text{Abs}\left[\int_0^1 e^{i\,(\text{nFZones}\,\pi)\,\rho^2}\, 2\,\rho\, d\rho\right]^2 = 1$$

$$\text{If nFZones} > 0, \qquad \text{Abs}\left[\int_0^1 e^{i\,(\text{nFZones}\,\pi)\,\rho^2}\, 2\,\rho\, d\rho\right]^2 = \frac{2 - 2\,\text{Cos}[\text{nFZones}\,\pi]}{\text{nFZones}^2\,\pi^2},$$

The following shows a plot of the on-axis irradiance as a function of number of Fresnel zones present.

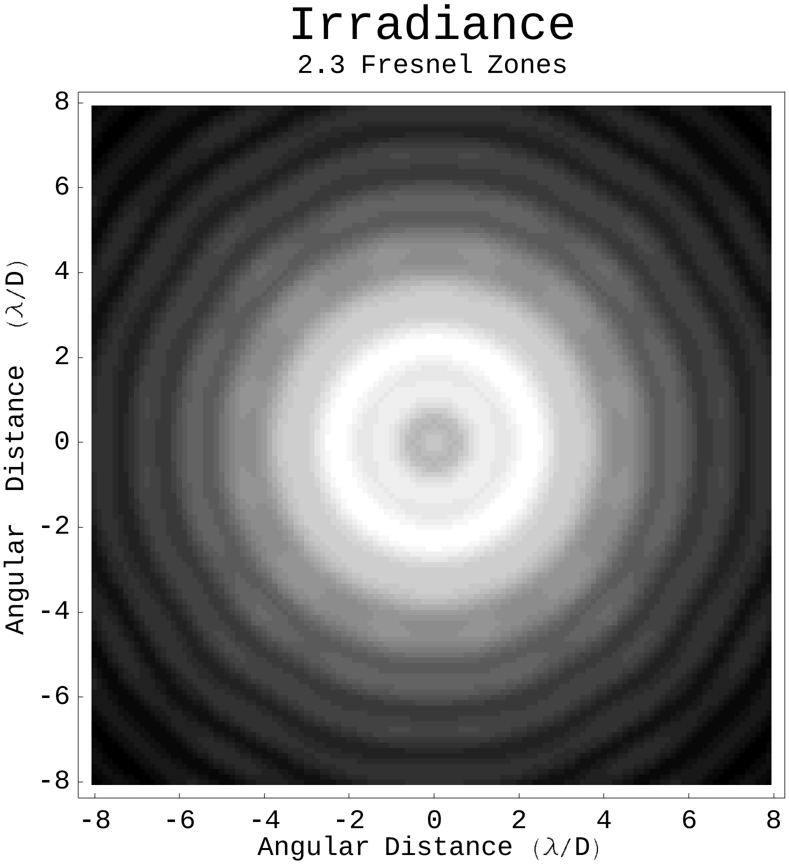


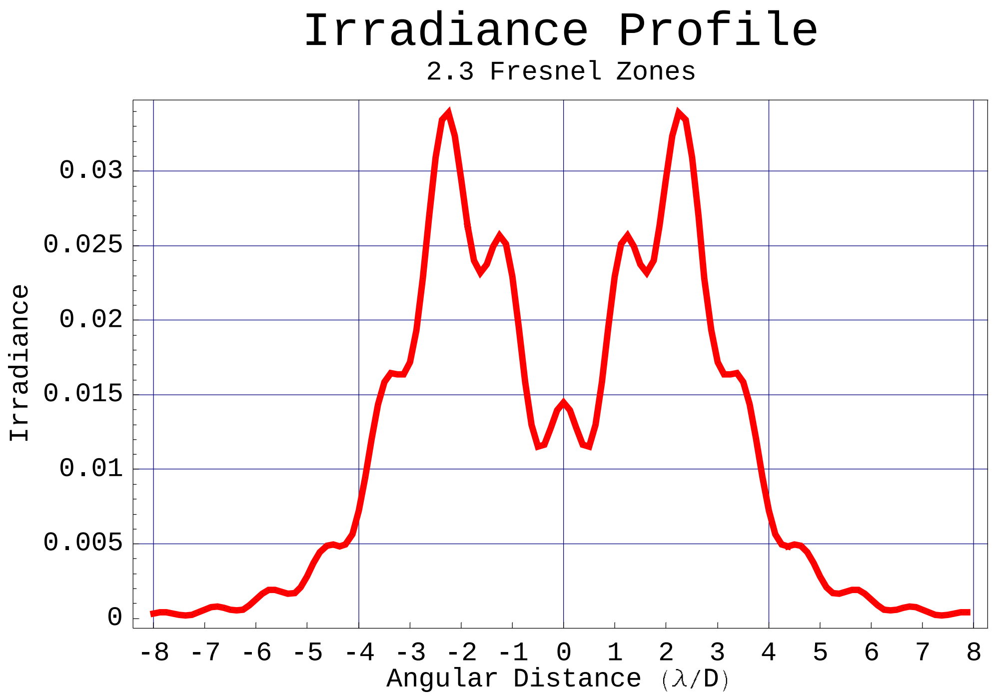
As can be seen the on-axis irradiance drops rapidly as the number of Fresnel zones within the aperture increases if the aperture size is held constant. This makes sense since the odd and even zone pairs cancel each other and as the number of zones increases the area of each zone decreases. A peak occurs when an odd number of Fresnel zones is present but the useful aperture area will go as 1/number of Fresnel zones and the maximum irradiance goes as 1/(π number of Fresnel zones)^2. This is different from the case where the area of the aperture is proportional to the number of zones present. In this latter case the odd and even zone pairs cancel, but the aperture area increases so an extra zone produces as much irradiance as the first zone would produce by itself.

Fourier transform

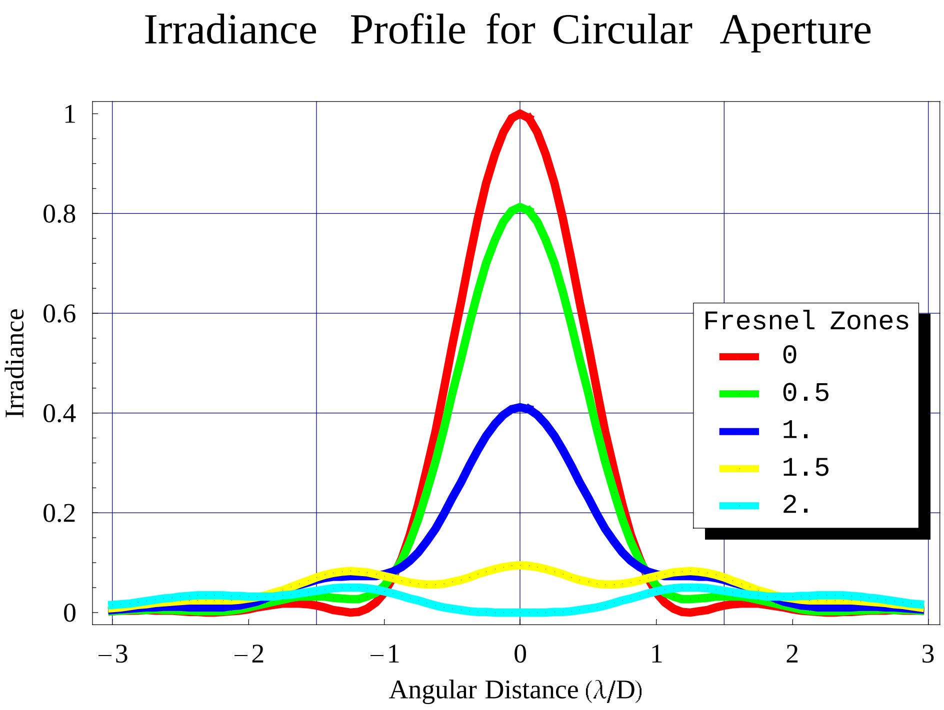
While the Hankel transform is a good technique for calculating the Fresnel diffraction pattern of a circular aperture, Fast Fourier transforms have become so fast on computers that for many calculations a Fast Fourier transform is a better approach for calculating Fresnel diffraction patterns than the Hankel transform. As shown in Eqn. 9 the amplitude of the Fresnel diffraction pattern of a circular aperture is obtained by taking the Fourier transform of $e^{i\,(\text{nFZones}\,\pi)\,(\xi^2+\eta^2)/r_{\text{max}}^2}$ where for simplicity a normalization factor is used so the nFZones=0 case (Fraunhofer diffraction) is normalized to 1.

The following is one example of the Fresnel diffraction calculated using a Fourier transform. In this example we will use a data array of 64 x 64 datapoints and then we will pad the array with zeroes to obtain a total array of 512 x 512.





Above we used Hankel transforms to calculate the irradiance profile for 0, 0.5, 1.0, 1.5, and 2 Fresnel zones in the aperture. Below we will use Fast Fourier transforms to do the same calculation. In this example we will use a data array of 64 x 64 datapoints and then we will pad the array with zeroes to obtain a total array of 1024 x 1024. The results for FFT's will be seen to be very similar to the results for the Hankel transforms and the calculation time is much shorter.



13.3 Circular Obstacle

One of the most amazing things in optics is the Poisson spot or as it is sometimes called the spot of Arago.

Let a small opaque circular disk be illuminated with a spherical wave and let u_{od} be the amplitude at the center of the diffraction pattern. At first thought one might think that $u_{od}[P]$ would be equal to zero, but that is not the case as shown below. If the disk is removed the unobstructed amplitude at point P is $u_{oo}[P]$. From Eq. 4 we know that if the opaque disk is replaced with a circular aperture of the same diameter as the opaque disk the on-axis amplitude would be $u_o[P] = u_{oo}[P]\,(1 - e^{i\pi\zeta})$. From Babinet's principle we know that

$u_{oo}[P] = u_{od}[P] + u_o[P]$ or

$u_{od}[P] = u_{oo}[P] - u_o[P] = u_{oo}[P]\,e^{i\pi\zeta}$

and for the on-axis irradiance

$I_{od}[p] = I_{oo}[p]$

That is, as long as ζ is not too large to make the normal Fresnel diffraction approximations invalid, there is a bright spot on axis at the center of the shadow that has the same irradiance as the unobstructed value of the irradiance. As the disk becomes larger the spot becomes smaller and contains less energy, but the on-axis irradiance remains equal to the unobstructed irradiance. Thus there is a bright spot everywhere along the central axis except immediately behind the circular obstacle. This is indeed an amazing result. If the eye is placed so as to collect light from the bright on-axis spot and one looks back at the opaque disk the edges of the disk will be bright.

If the source were an extended source each point on the source would form a bright spot at the center of the shadow it produces. Thus a small disk acts as a crude positive lens.

13.4 Fresnel Zone Plate

In discussing the Fresnel diffraction of a circular aperture we utilized the fact that successive Fresnel zones nullify each other. This suggests that we will observe a tremendous increase in irradiance at the on-axis point P if we remove all the odd or all the even zones. A screen that alters the light, either in amplitude or phase, coming from every other half-period zone is called a zone plate.

Binary Amplitude Fresnel Zone Plate

Figure 2 is an example of a Fresnel zone plate where every other Fresnel zone is blocked.

From Eq. 1 the radius of the nth Fresnel zone is given by

$$\rho_n = \left(\frac{\lambda\, z_1\, z_2\, n}{z_1 + z_2}\right)^{1/2} \qquad (13)$$

The amplitude at on-axis point P can be written as

$$\begin{aligned} U[P] &= U_1 + U_3 + U_5 + \cdots + U_N \\ &= \frac{\pi\, u_{oo}[P]}{i} \left(\int_0^1 e^{i\pi q}\, dq + \int_2^3 e^{i\pi q}\, dq + \int_4^5 e^{i\pi q}\, dq + \cdots + \int_{N-1}^N e^{i\pi q}\, dq \right) \\ &= u_{oo}[p]\,(N+1) \end{aligned}$$

$$I[P] = (N+1)^2\, I_{oo}[P] \tag{14}$$

Thus we get a very intense spot of light.

If we keep λ, z_1, and and the size of the zone plate constant, but vary z_2, the limits on the integrals will no longer be integers. The integrals will give complex numbers whose magnitudes are less than $2/\pi$. Therefore, the zone plate has focal properties like for a lens and in this respect differs from a simple opaque disk. The focal length of the zone plate follows from Eq. 13.

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{n\lambda}{\rho_1{}^2}$$

Letting n =1,

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{\lambda}{\rho_1{}^2} \tag{15}$$

The zone plate behaves like a lens where the focal length is given by $f = \frac{\rho_1{}^2}{\lambda}$. This is a focal length that is inversely proportional to the wavelength. Likewise, n could be -1 and the focal length would be $-\frac{\rho_1{}^2}{\lambda}$.

z_2 could be changed so each clear portion of the zone plate contains a multiple number of Fresnel zones. If the clear portion contains an even number of Fresnel zones the Fresnel zones will cancel one another. However, if an odd number of Fresnel zones are contained one of the zones will not be cancelled and a fainter focus will be obtained. For example if 3 zones are present, the effects of two of the Fresnel zones will cancel, but that of the third is left over. In this case the amplitude for the third-order will be 1/3 that of the first order and the irradiance will be 1/9 that of the first order. Thus we have fainter images for focal lengths of

$$f = \pm\frac{\rho_1{}^2}{3\lambda},\ \pm\frac{\rho_1{}^2}{5\lambda},\ \pm\frac{\rho_1{}^2}{7\lambda},\ \cdots$$

The focal lengths and image irradiance are the same for transparent odd Fresnel zones as for transparent even Fresnel zones, but the phase of the image will differ by 180° for the two cases.

Binary Phase Fresnel Zone Plate

Instead of having every other Fresnel zone opaque a Fresnel zone plate can be made by having all Fresnel zones transparent, but having a π phase step on every other Fresnel zones. In this way all the Fresnel zones contribute in phase and the resulting amplitude is twice that of a binary amplitude (every other zone blocked) Fresnel zone plate. Since the amplitude increases by a factor of 2, the irradiance increases by a factor of 4.

Polarization Fresnel Zone Plate

An interesting Fresnel zone plate is one where vertical polarizers cover the odd zones and horizontal polarizers cover the even zones. In this situation the odd and even zones operate independently since the polarization of the light transmitted through them is orthogonal. There are three interesting situations to look at.

- **Incident radiation is horizontally or vertically polarized**

Only even or odd Fresnel zones transmit radiation and image irradiance and behavior is identical to that of regular binary amplitude zone plate.

- **Incident radiation unpolarized**

Even and odd Fresnel zones each transmit half of the incident radiation and the image irradiance and behavior is identical to that of regular binary amplitude zone plate.

- **Incident radiation linearly polarized at angle θ relative to horizontal**

Let the incident radiation have amplitude E then E Cos[θ] is transmitted through even zones and E Sin[θ] is transmitted through odd zones. The resultant irradiance goes as $E^2\,\mathrm{Cos}[\theta]^2 + E^2\,\mathrm{Sin}[\theta]^2 = E^2$. Thus, the image irradiance and behavior is identical to that of regular binary amplitude zone plate.

The interesting result is that independent of the polarization of the incident light the amount of light in the orders is the same as for a conventional binary amplitude Fresnel zone plate.

13.5 Rectangular Aperture

Next we will look at the Fresnel diffraction of a rectangular aperture as shown in Figure 5. It follows from Eq 8 that if the source is a point source of amplitude A at coordinates (x’, y’) a distance z_1 to the left of the aperture and Pupil[ξ, η] is the aperture amplitude transmittance function

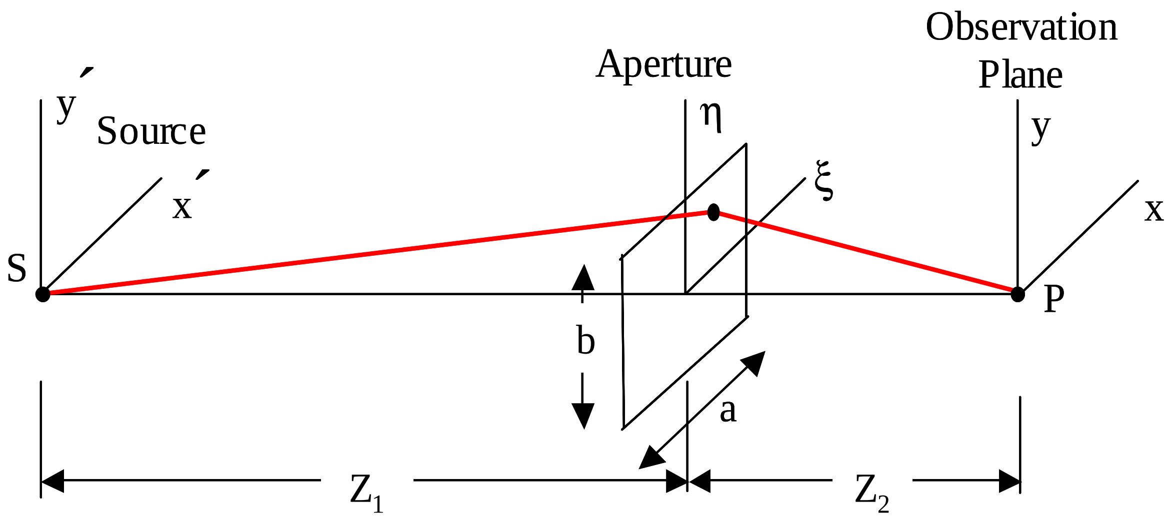


Fig. 5. Fresnel diffraction of rectangular aperture.

$$u[x,y] = \frac{A\, e^{i k (z_1+z_2)}}{i \lambda z_1 z_2} \int_{-\infty}^{\infty} \text{Pupil}[\xi,\eta]\, e^{i \frac{k}{2 z_2} ((x-\xi)^2+(y-\eta)^2)}\, e^{i \frac{k}{2 z_1} ((x'-\xi)^2+(y'-\eta)^2)}\, d\xi\, d\eta \tag{16}$$

If the illuminating source is on axis so x’ = y’ = 0, then we get Eqn. 9 which is repeated below.

$$u[x,y] = \frac{A\, e^{i k (z_1+z_2)}}{i \lambda z_1 z_2}\, e^{i \frac{\pi}{\lambda z_2} (x^2+y^2)} \int_{-\infty}^{\infty} \left(\text{Pupil}[\xi,\eta]\, e^{i \frac{\pi}{\lambda} \left(\frac{1}{z_1}+\frac{1}{z_2}\right) (\xi^2+\eta^2)}\right) e^{-i \frac{2\pi}{\lambda z_2} (x \xi + y \eta)}\, d\xi\, d\eta \tag{17}$$

Rather than finding the irradiance at various points of the observing screen with the aperture fixed, the normal way of solving for the Fresnel diffraction pattern of a rectangular aperture is to fix the observation point, P, and cause the pattern to sweep over it by translating the aperture in its own plane. The two approaches are not strictly equivalent since the pattern on the screen changes both in size and distribution of irradiance as the aperture is moved. These changes are negligible if the aperture motion is small compared to z_1. The following three cases can be studied without loss of generality.

1) If a and b are small compared to z_1 the entire pattern can be examined.
2) If a is small and b is large, the pattern can be investigated as the aperture is translated parallel to the ξ axis.
3) If a and b are both large the pattern can be investigated near the geometrical shadow. (This is the case of the semi-infinite screen with a straight edge.)

Letting x and y equal zero yields

$$\begin{aligned} u[P] &= \frac{A\, e^{i k (z_1+z_2)}}{i \lambda z_1 z_2} \int_{-\xi_1}^{\xi_2}\int_{-\eta_1}^{\eta_2} e^{i \frac{\pi}{\lambda} \left(\frac{1}{z_1}+\frac{1}{z_2}\right) (\xi^2+\eta^2)}\, d\xi\, d\eta \\ &= \frac{A\, e^{i k (z_1+z_2)}}{i \lambda z_1 z_2} \int_{-\xi_1}^{\xi_2}\int_{-\eta_1}^{\eta_2} e^{i \frac{\pi}{\lambda} \left(\frac{z_1+z_2}{z_1 z_2}\right) (\xi^2+\eta^2)}\, d\xi\, d\eta \end{aligned} \tag{18}$$

Making a substitution of variables of

$$u = \eta \sqrt{\frac{2\,(z_1 + z_2)}{\lambda\, z_1\, z_2}} \quad \text{and} \qquad v = \xi \sqrt{\frac{2\,(z_1 + z_2)}{\lambda\, z_1\, z_2}} \quad \text{yields}$$

$$u[P] = \frac{A\, e^{i\, k\, (z_1 + z_2)}}{2\, i\, (z_1 + z_2)} \int_{u_1}^{u_2} e^{i\, \pi\, u^2/2}\, du \int_{v_1}^{v_2} e^{i\, \pi\, v^2/2}\, dv$$

$$\text{But} \quad \frac{A\, e^{i\, k\, (z_1 + z_2)}}{2\, i\, (z_1 + z_2)} = \frac{u_{oo}[P]}{2\, i}$$

where $u_{oo}[P]$ is the unobstructed amplitude at point P. Therefore, we can write

$$u[P] = \frac{u_{oo}[P]}{2\, i} \int_{u_1}^{u_2} e^{i\, \pi\, u^2/2}\, du \int_{v_1}^{v_2} e^{i\, \pi\, v^2/2}\, dv$$

These integrals are not elementary but they can be written in terms of the Fresnel integrals

$$C[u] = \int_0^u \mathrm{Cos}\left[\frac{\pi\, t^2}{2}\right] dt \quad \text{and} \quad S[u] = \int_0^u \mathrm{Sin}\left[\frac{\pi\, t^2}{2}\right] dt$$

$$u[P] = \frac{u_{oo}[P]}{2\, i}\, (C[u_2] - C[u_1] + i\, (S[u_2] - S[u_1]))\, (C[v_2] - C[v_1] + i\, (S[v_2] - S[v_1]))$$

$$= \frac{u_{oo}[P]}{2\, i}\, a\, e^{i\, \psi}\, b\, e^{i\, \chi}$$

Then

$$i[P] = \frac{1}{4}\, i_{oo}[P]\, a^2\, b^2$$

Effect of translating aperture sideways

In calculating the effects of translating the aperture sideways it is convenient to introduce some new parameters. Let b be the aperture width and η_o be the coordinate of the center of the aperture. Then

$$\Delta u = u_2 - u_1 = b \left(\frac{2\,(z_1 + z_2)}{\lambda\, z_1\, z_2}\right)^{1/2} \quad \text{and}$$

$$u_o = \frac{1}{2}\, (u_2 + u_1) = \eta_o \left(\frac{2\,(z_1 + z_2)}{\lambda\, z_1\, z_2}\right)^{1/2}$$

The integral will now have the limits $u_o - \frac{\Delta u}{2}$ and $u_o + \frac{\Delta u}{2}$. By similar triangles we see the if the aperture is displaced a distance η_o sideways the corresponding point in the observation plane is

$$y_o = \eta_o \left(\frac{z_1 + z_2}{z_1}\right)$$

If the aperture is illuminated with collimated light $y_o = \eta_o$.

Fresnel number for rectangular aperture

It can be instructive to determine a relationship between the Fresnel number of a rectangular aperture and Δu. If the aperture width is b the Fresnel number, nFZones, is given by

$$\left(\frac{b}{2}\right)^2 \left(\frac{1}{2\,z_1} + \frac{1}{2\,z_2}\right) = \text{nFZones}\,\frac{\lambda}{2}$$

$$\text{nFZones} = \frac{\frac{b^2}{8}\left(\frac{z_1+z_2}{z_1\,z_2}\right)}{\lambda/2} = \frac{1}{8}\,b^2\left(\frac{2\,(z_1+z_2)}{\lambda\,z_1\,z_2}\right) = \frac{(\Delta u)^2}{8}$$

$$\text{nFZones} = \frac{(\Delta u)^2}{8}$$

Plots of Fresnel diffraction of rectangular aperture of width b

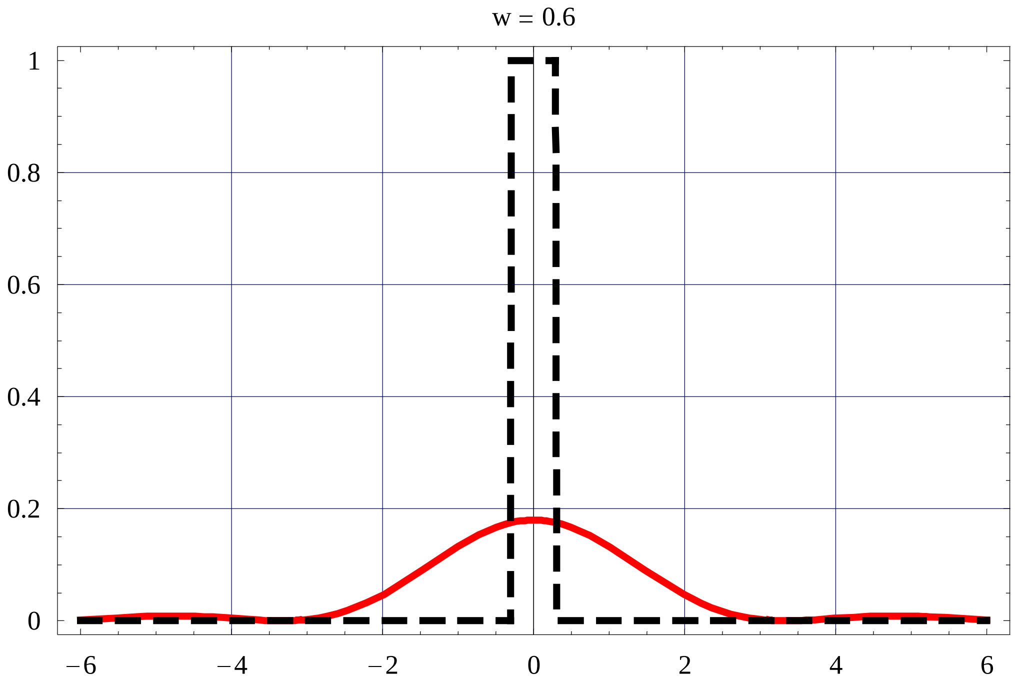
We will now make some plots of the Fresnel diffraction of a rectangular aperture of width b .

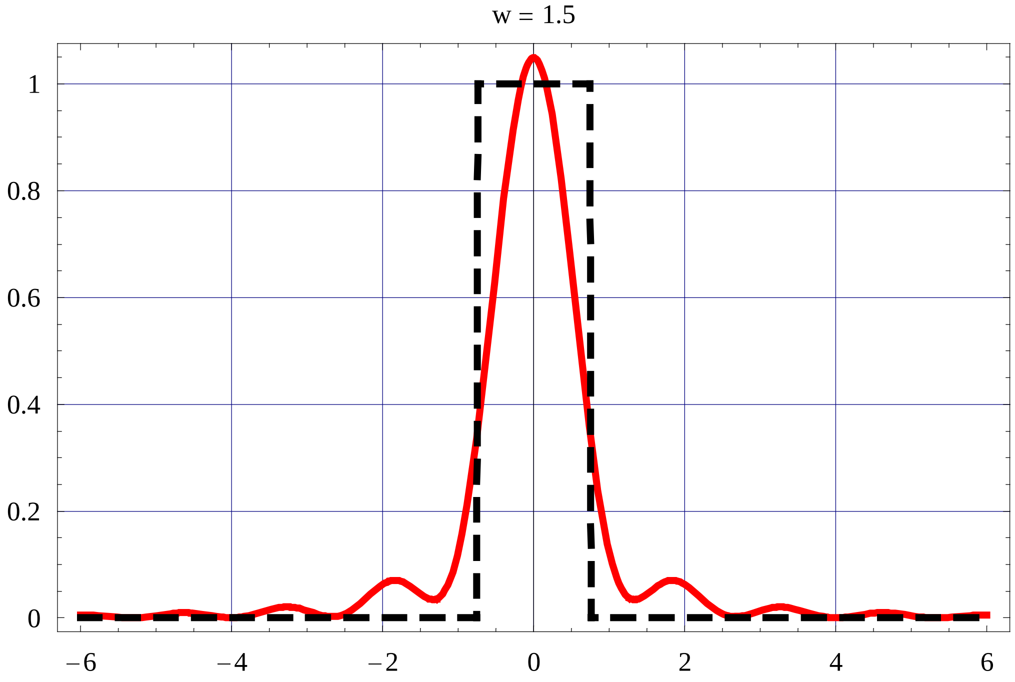
From above, the width of the geometrical shadow will be $w = \Delta u = b\left(\frac{2\,(z_1+z_2)}{\lambda\,z_1\,z_2}\right)^{1/2}$. As will be seen below, when w is greater than 10 (nfzones ≈ 12), the pattern is roughly that predicted by geometrical optics, the principal difference being the prominent fringes near the edge of the pattern. As the aperture shrinks, the pattern shrinks roughly in proportion, but the system of fringes becomes more complicated. When w passes a value of about 2 (nfzones ≈ 1/2), the pattern ceases to shrink and begins to expand. When w is less than unity (1/8 Fresnel zones), the pattern shows roughly the profile characteristic of Fraunhofer diffraction. Note that when w is greater than unity the irradiance in the bright region hovers about the unobstructed irradiance, but after w becomes less than unity the maximum irradiance falls rapidly because a decreasing amount of power is being spread over an increasing area.

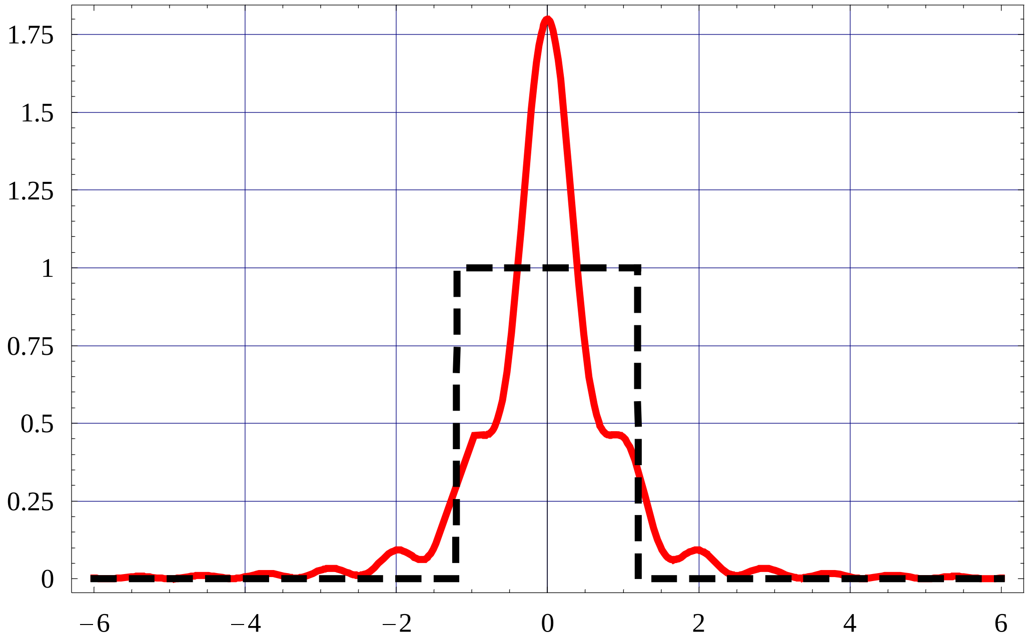
The *Mathematica* equations to be used in the calculations are

```
amplitude[u1_, u2_] :=
  1
 ---- (FresnelC[u2] - FresnelC[u1] + ⅈ (FresnelS[u2] - FresnelS[u1]))
 √2
irradiance[u1_, u2_] := (Abs[amplitude[u1, u2]])^2
```

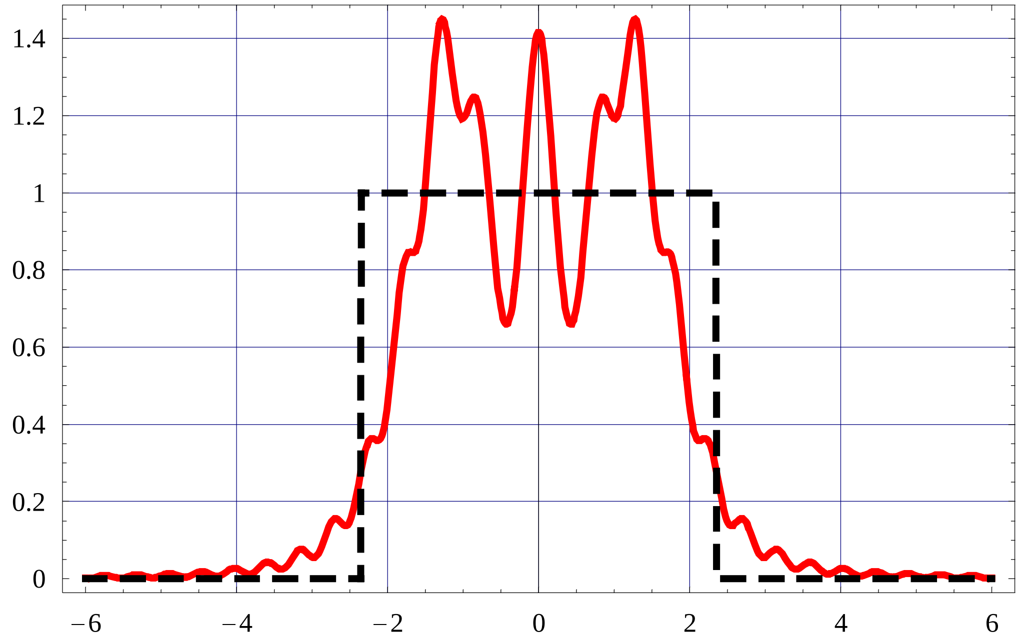
The dashed line in the following plots shows the geometrical shadow.

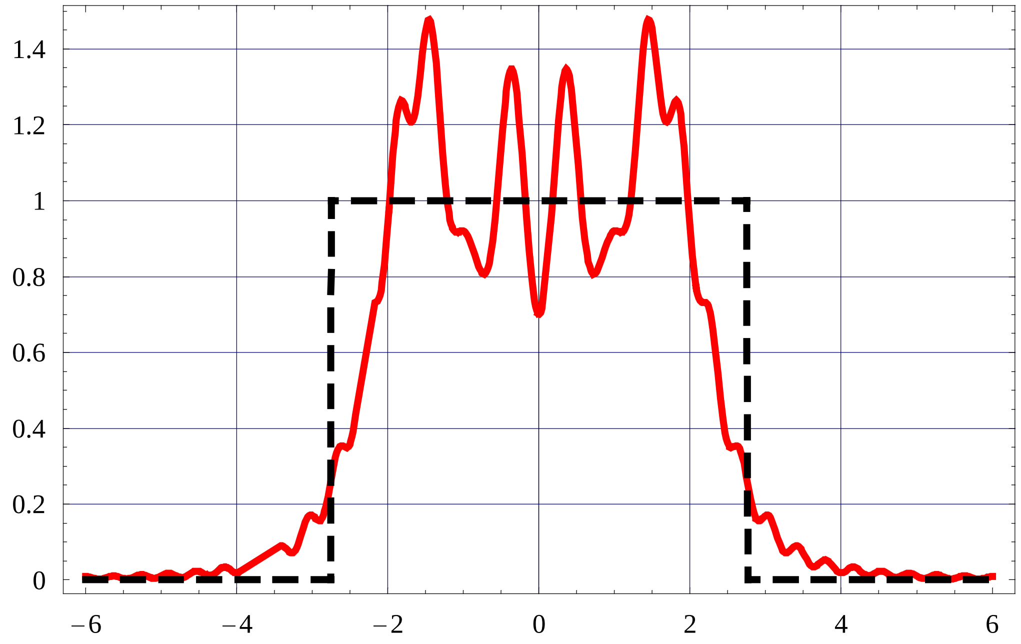

w = 0.6
1
0.8
0.6
0.4
0.2
0
−6
−4
−2
0
2
4
6

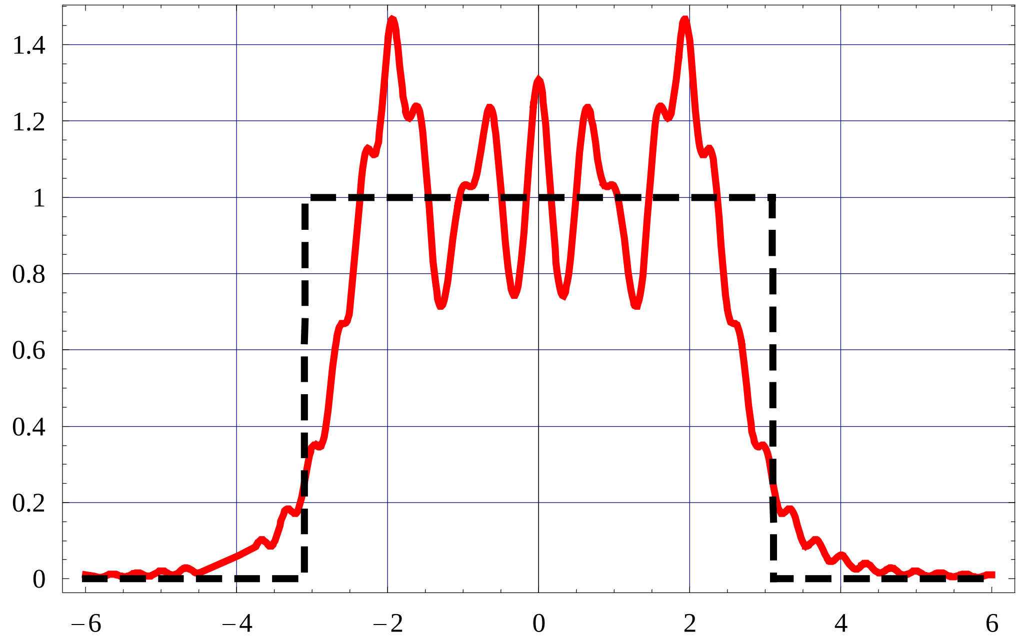

w = 1.5
1
0.8
0.6
0.4
0.2
0
−6
−4
−2
0
2
4
6

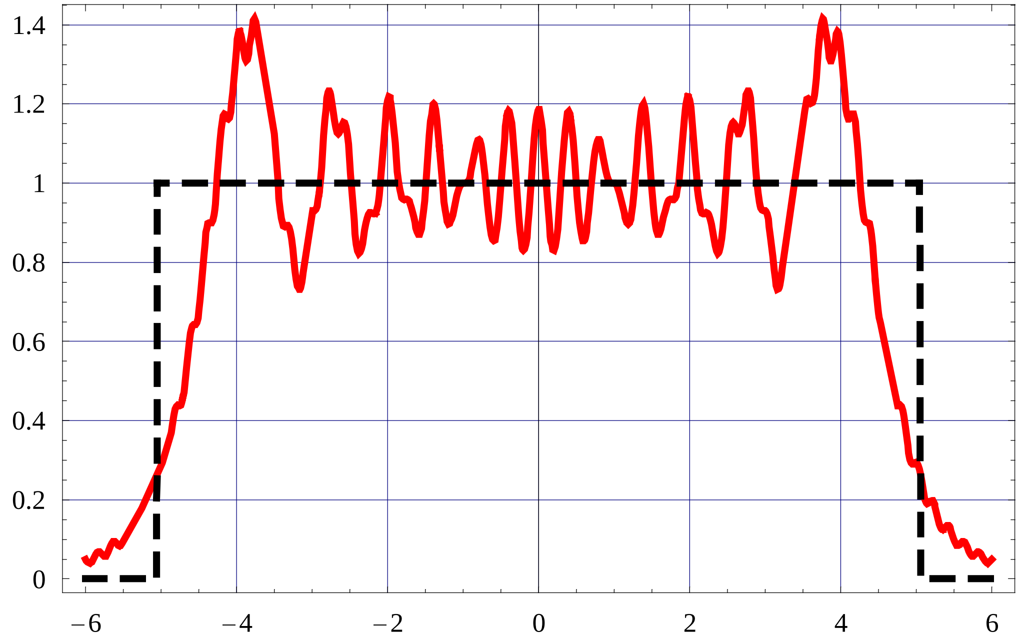

w = 2.4


w = 3.7


w = 4.7


w = 5.5


w = 6.2


w = 10.1

Fresnel diffraction for double slits

The same approach described above for finding the Fresnel diffraction of a single slit can be used to find the Fresnel diffraction of a double slit. As an example we will calculate the Fresnel diffraction pattern of the double slits shown in Figure 6 below. Let the slits have a width D and a separation L. The source S is at infinity and the observation plane is at a distance F=1,000,000 wavelengths. D=1000 wavelengths and L=3000 wavelengths. To complicate the situation let the lower slit have a retardation of 1/2 wavelength. The diffraction pattern is calculated as follows.

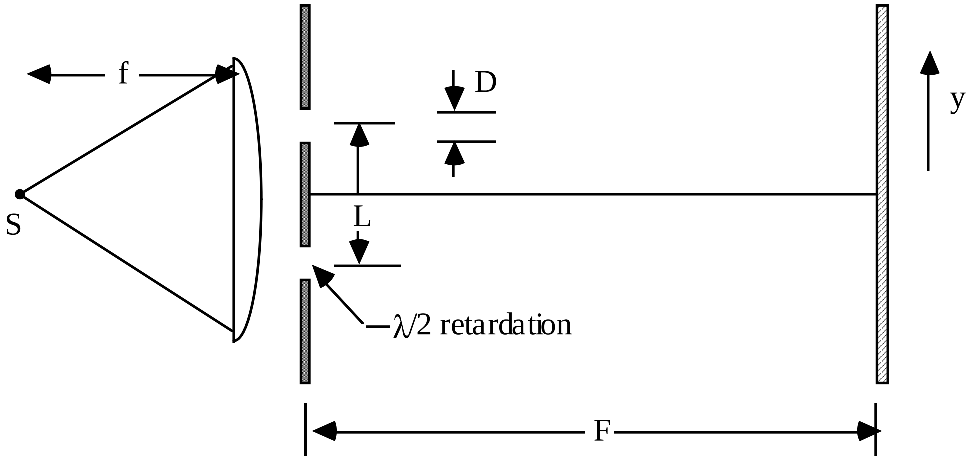


Fig. 6. Double slit configuration.

Remember from above that

$$u = \eta \sqrt{\frac{2\,(z_1 + z_2)}{\lambda\, z_1\, z_2}}$$

Since the light illuminating the double slits is collimated we can write

$$u = y\left(\frac{2}{\lambda F}\right)^{1/2} = y\left(\frac{2}{\lambda\,(1,000,000\,\lambda)}\right)^{1/2} = \frac{y\sqrt{2}}{1000\,\lambda}$$

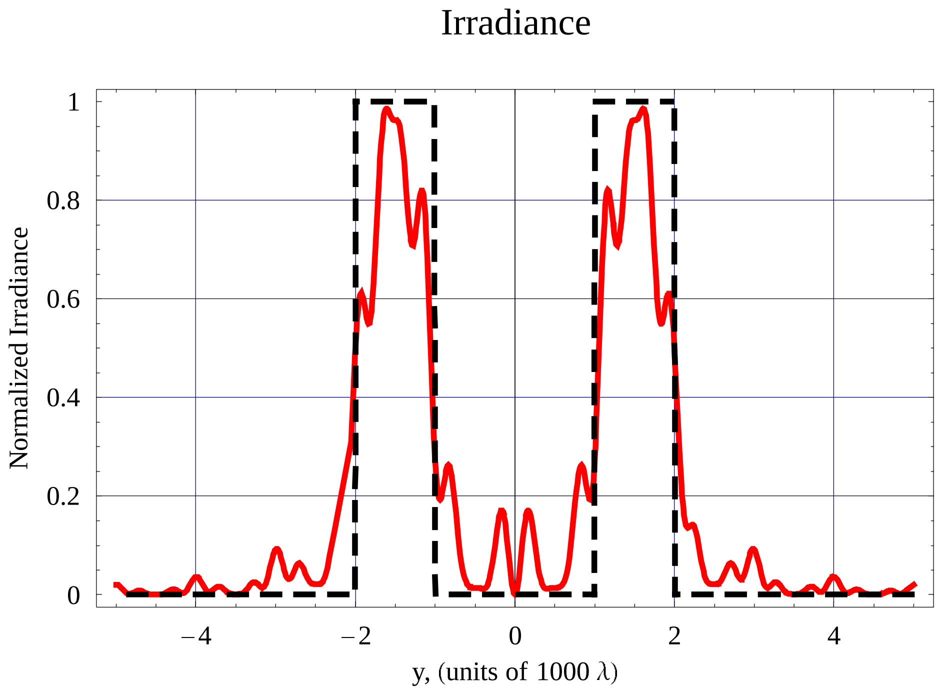
`Top Slit`

$$u_1 = \frac{(y - 2000\,\lambda)\,\sqrt{2}}{1000\,\lambda}, \quad u_2 = \frac{(y - 1000\,\lambda)\,\sqrt{2}}{1000\,\lambda}$$

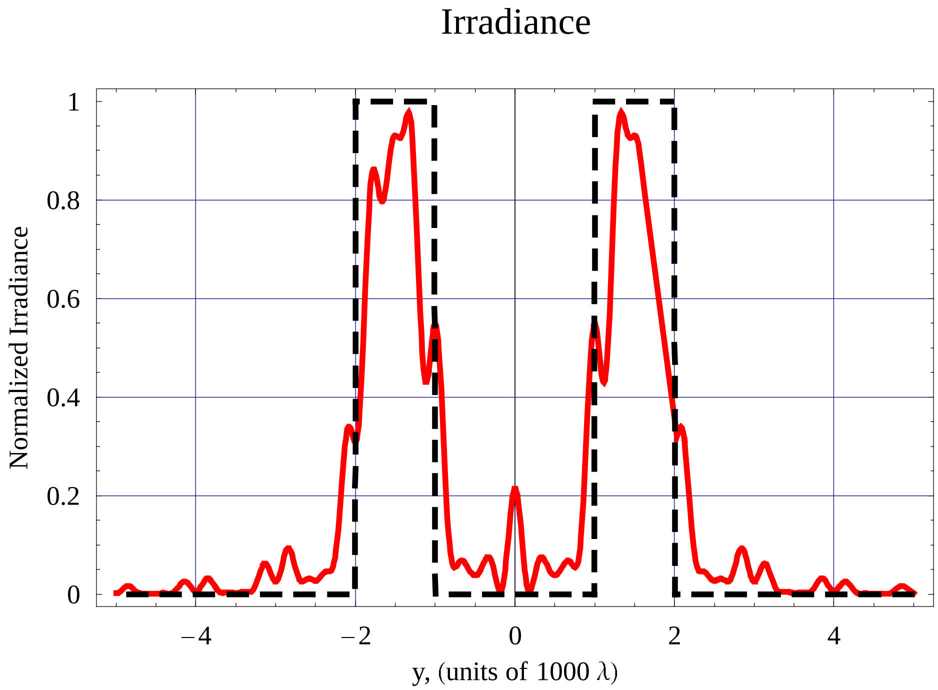
`Bottom Slit`

$$u_1 = \frac{(y + 1000\,\lambda)\,\sqrt{2}}{1000\,\lambda}, \quad u_2 = \frac{(y + 2000\,\lambda)\,\sqrt{2}}{1000\,\lambda}$$

The dashed line in the plot shows the geometrical shadow.



If the $\lambda/2$ retardation plate is not present the amplitudes due to the two slits add and we get



Large aperture - Long slit and straight edge

If the dimensions of the aperture, a and b, both become large, approximations concerning the obliquity factor, $\frac{1}{r_1 r_2}$, and $e^{i(r_1+r_2)}$ breakdown. A more exact solution is rather difficult, one problem being that the double integral does not break down into a product of simple integrals. However, if we ignore this problem we end up with

$$\begin{aligned} a\, e^{i\psi} &= C[\infty] - C[-\infty] + i\,(S[\infty] - S[-\infty]) \\ &= \sqrt{2}\, e^{i\pi/4} \end{aligned}$$

Likewise $b\, e^{i\chi} = \sqrt{2}\, e^{i\pi/4}$

Therefore,

$u[P] = \frac{u_{oo}[P]}{2 i}\, a\, e^{i\psi}\, b\, e^{i\chi}$ and $u[P] = u_{oo}[P]$

$i[P] = i_{oo}[P]$ as required.

It is not surprising we have the correct result since the fundamental restriction in the Kirchhoff theory was that z_1 and $z_2 >> \lambda$.

W will now look at a semi-infinite diffracting screen with a straight edge as shown in Fig. 6. Points in the ξ-η plane for $\eta > \eta_1$ will be covered and we will assume $|\eta_1| << z_1$ so the observation point is always near the edge of the geometrical shadow.

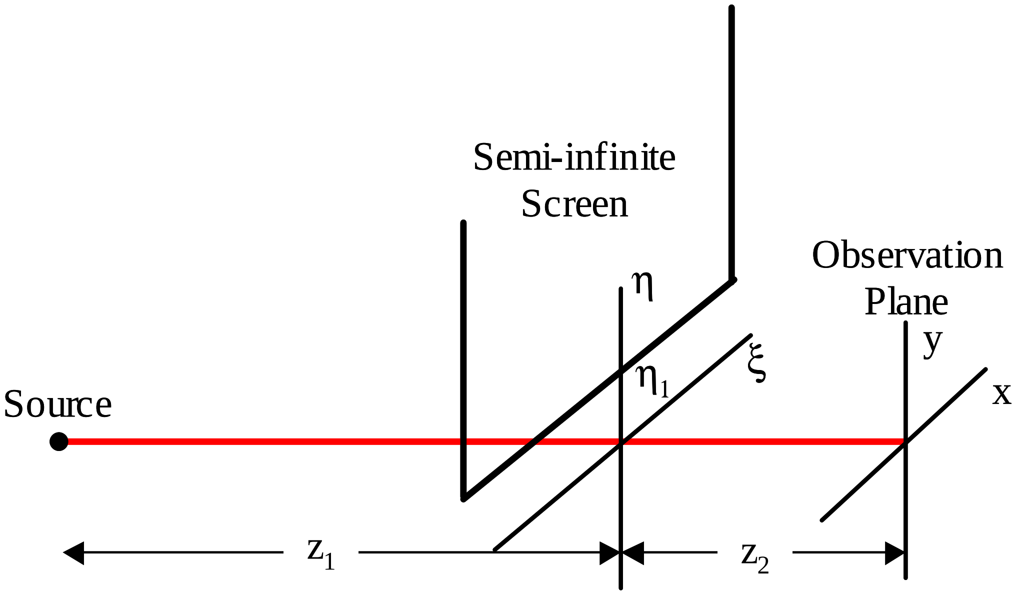


Fig. 6. Semi-infinite diffracting screen.

$\eta_2 = -\infty, \;\; \xi_1 = \infty, \;\; \xi = -\infty$

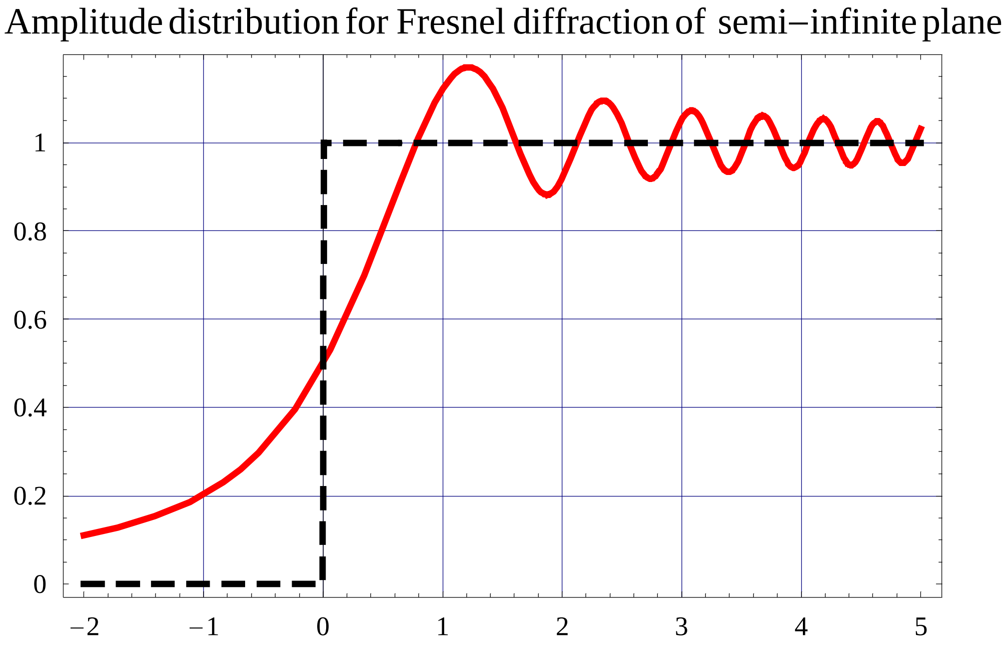
$\mathbf{a}\, e^{i\psi} = \mathbf{C}[\eta_1] - \mathbf{C}[-\infty] + i\,(\mathbf{S}[\eta_1] - \mathbf{S}[-\infty])$

$\eta_1 = 0$ corresponds to the observation point being at the edge of the geometrical shadow. As shown below, at this point the amplitude is $\frac{1}{2}$ and the irradiance is $\frac{1}{4}$ the unobstructed irradiance.

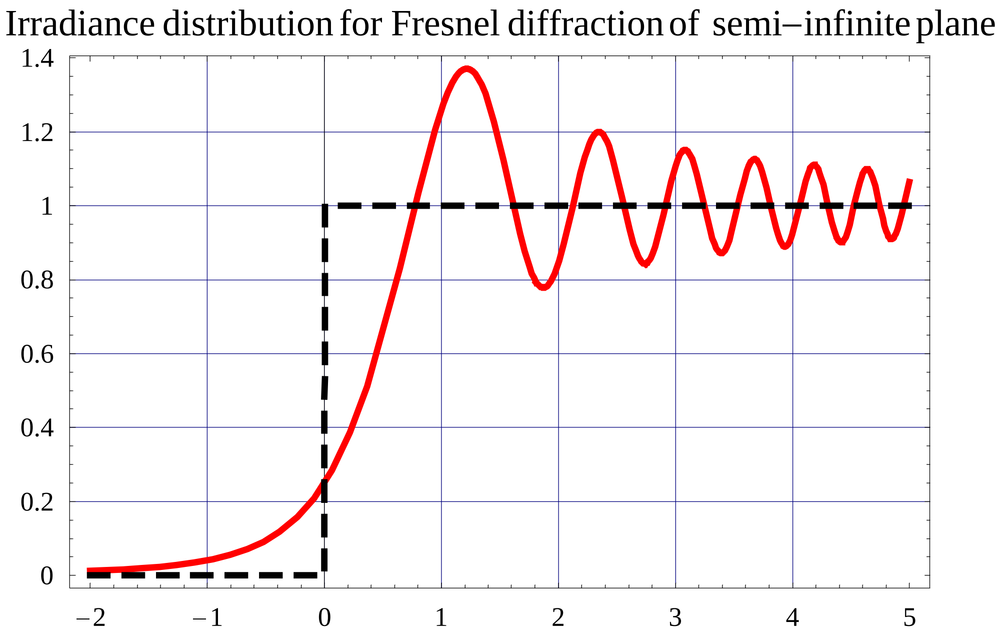
Examples are shown below for the diffraction pattern for a semi-infinite plane. The dashed line shows the geometrical shadow

Fresnel Diffraction pattern for semi-infinite plane

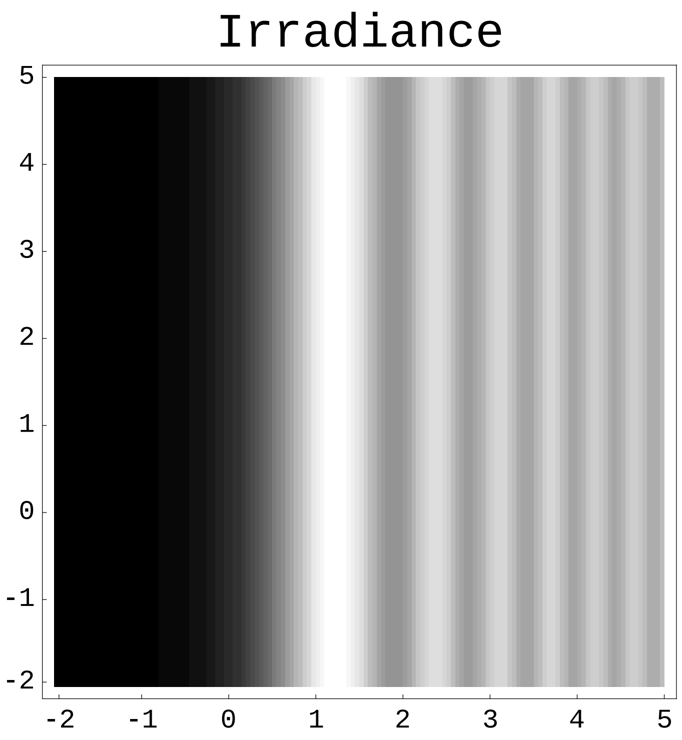
■ Calculate amplitude pattern for semi-infinite plane



- **Calculate irradiance pattern for semi-infinite plane**



- **Grey scale irradiance plot**



Rectangular obstacle

The Fresnel diffraction pattern for a rectangular obstacle can be calculated using Babinet's principle. Let

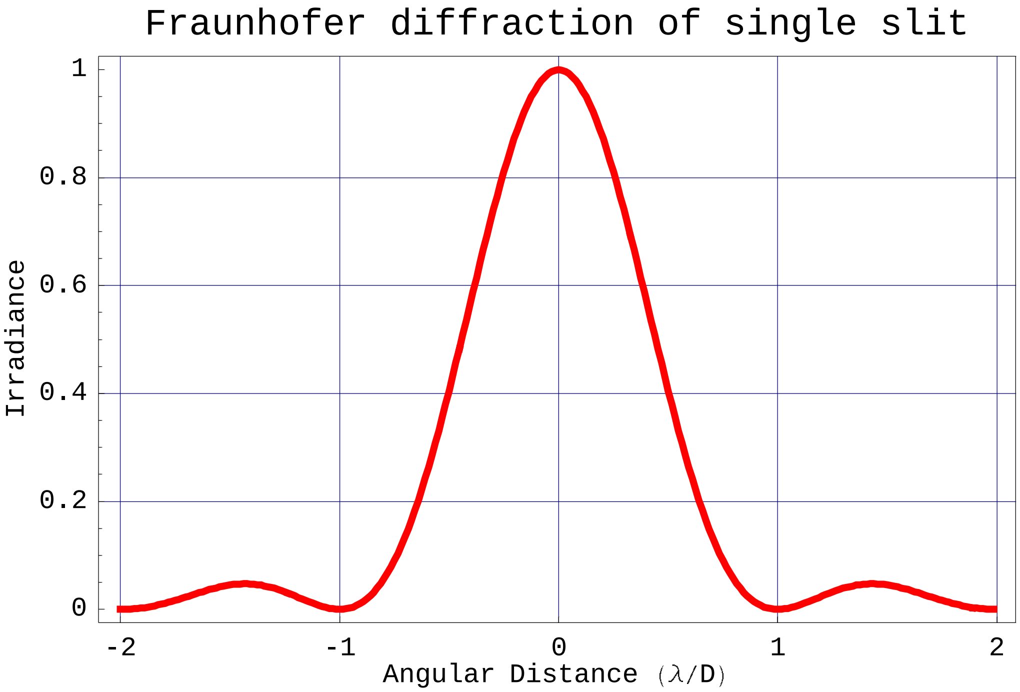
u_{slit} = amplitude for slit
u_{obstacle} = amplitude for obstacle
u_{oo} = amplitude for no obscuration, then
$u_{\text{obstacle}} = u_{\text{oo}} - u_{\text{slit}}$

Use of Fast Fourier transform to solve Fresnel diffraction of rectangular slit

As shown in Eq. 9 the Fresnel diffraction pattern of a rectangular slit can be thought of as the Fourier transform of

$$\text{Pupil}[\xi, \eta]\, e^{i \frac{\pi}{\lambda} \left(\frac{1}{z_1} + \frac{1}{z_2}\right) (\xi^2 + \eta^2)}$$

Since fast Fourier transforms are so fast it makes sense to calculate the Fresnel diffraction pattern of rectangular apertures using Fast Fourier transforms instead of using the normal approach of using Fresnel integrals. We will show some results below. For our example we will look at one-dimensional apertures starting with a Fraunhofer diffraction pattern.



We will now calculate the Fresnel diffraction pattern for rectangular of different Fresnel numbers. Before performing the calculation we want to determine a method for calculating the geometrical shadow of the rectangular aperture

Geometrical shadow for Fresnel diffraction of rectangular aperture

Let nFZones= number of Fresnel zones, then

$$\text{OPD} = \text{nFZones}\,\frac{\lambda}{2}$$

The OPD as a function of radius ρ can be written as

$$\text{OPD} = \text{nFZones}\,\frac{\lambda}{2}\,\rho^2\text{, then}$$

$$\text{slope} = \text{nfzones}\,\frac{\lambda}{2}\,2\,\rho = \text{nFZones}\,\lambda\,\rho$$

Slope in units of $\frac{\text{waves}}{\text{diameter}}$ or $\frac{\text{waves}}{\text{aperture width}} = \frac{\lambda}{b}$

$$\text{slopeWavesPerDiameter} = 2\,\text{nFZones}$$

Therefore the width of the geometrical shadow in units of waves per diameter is ± 2 nfzones

But

$$(\Delta\text{u})^2 = 8\,\text{nFZones}$$

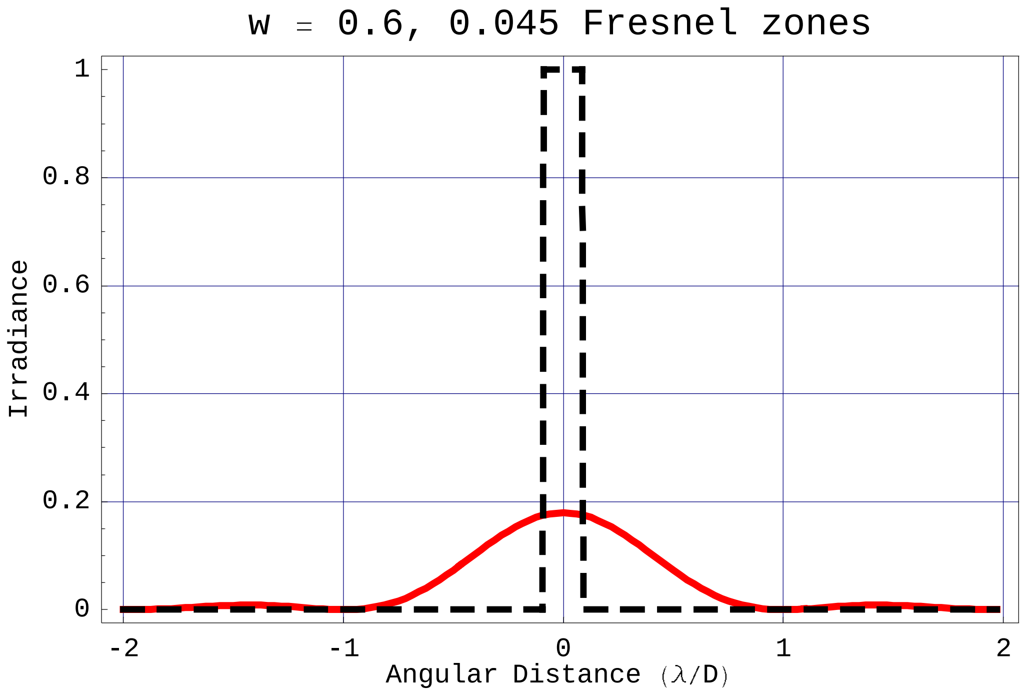
So in terms of $(\Delta\text{u})^2$ the width of the geometrical shadow is

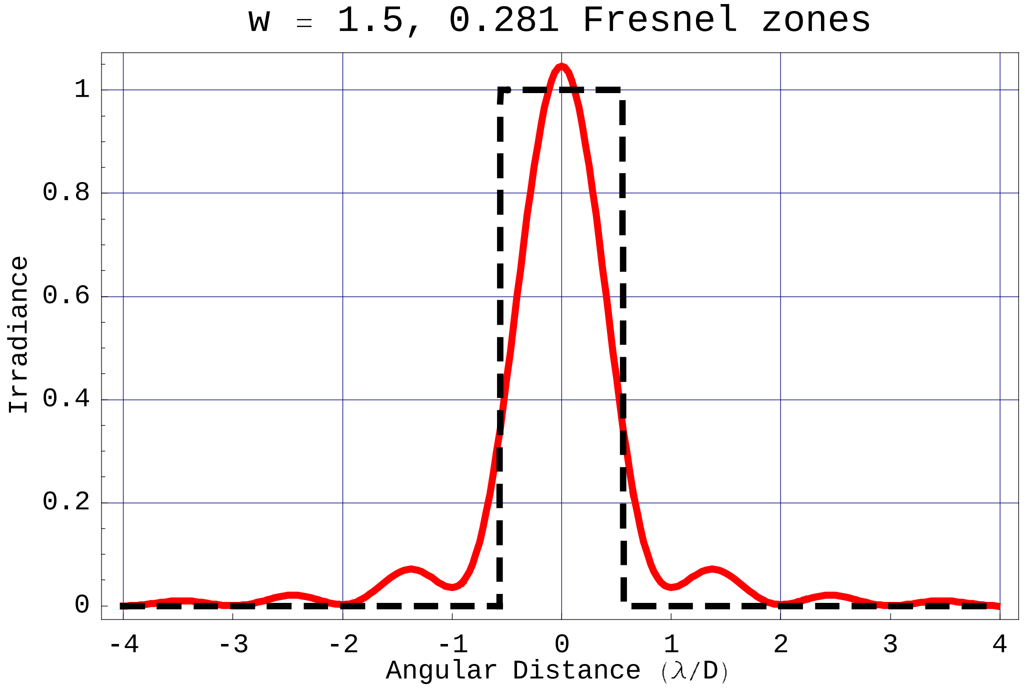
$$\pm\frac{(\Delta\text{u})^2}{4}\,\text{waves}/\text{radius}$$

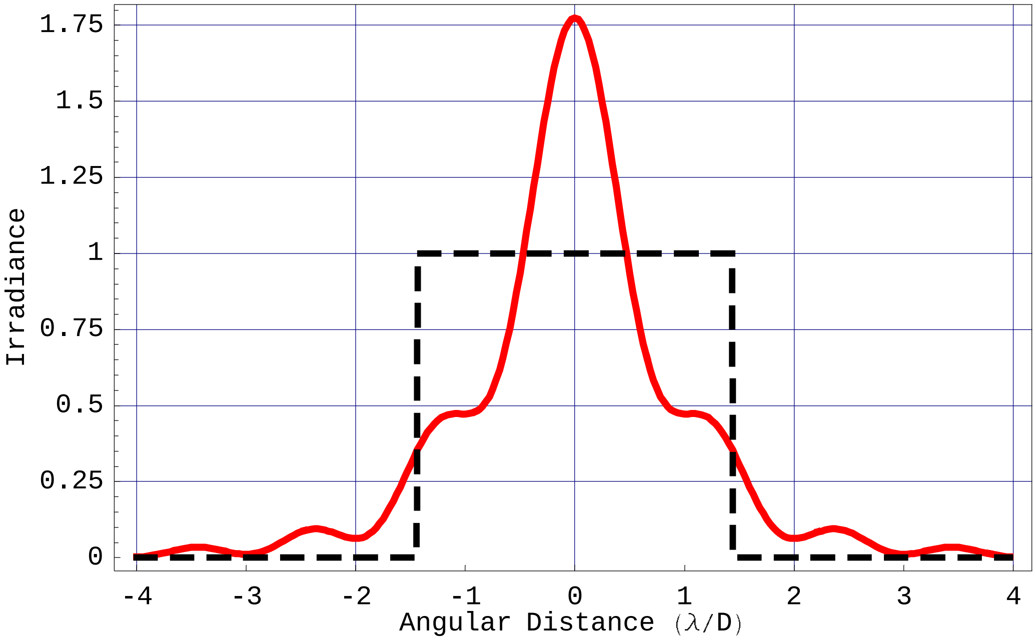
In terms of pixels in the Fourier transform the width of the geometrical shadow is

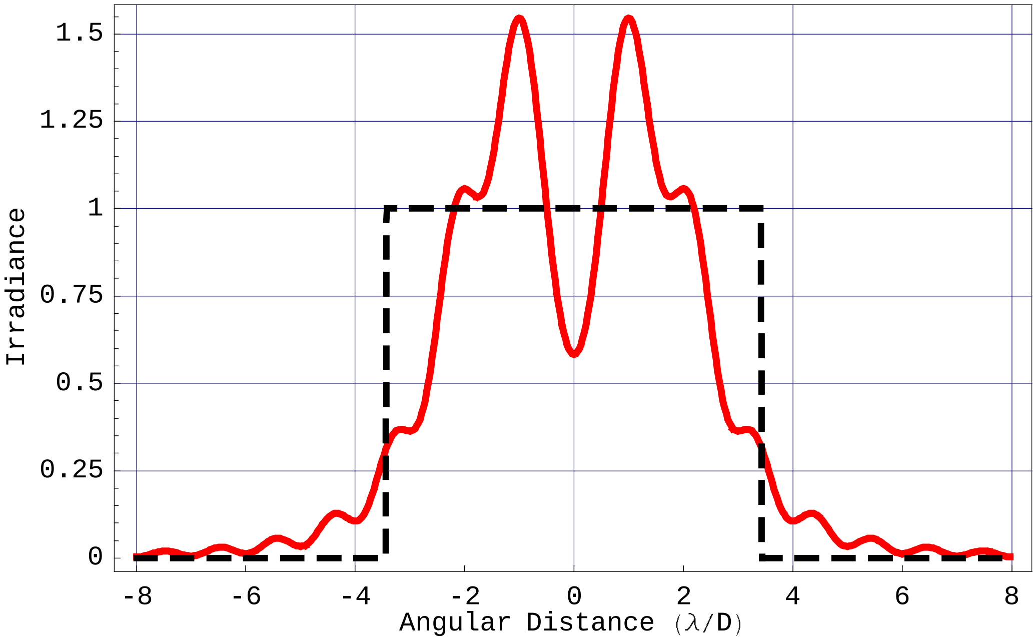
$$\pm\frac{(\Delta\text{u})^2}{4}\,\text{range}$$

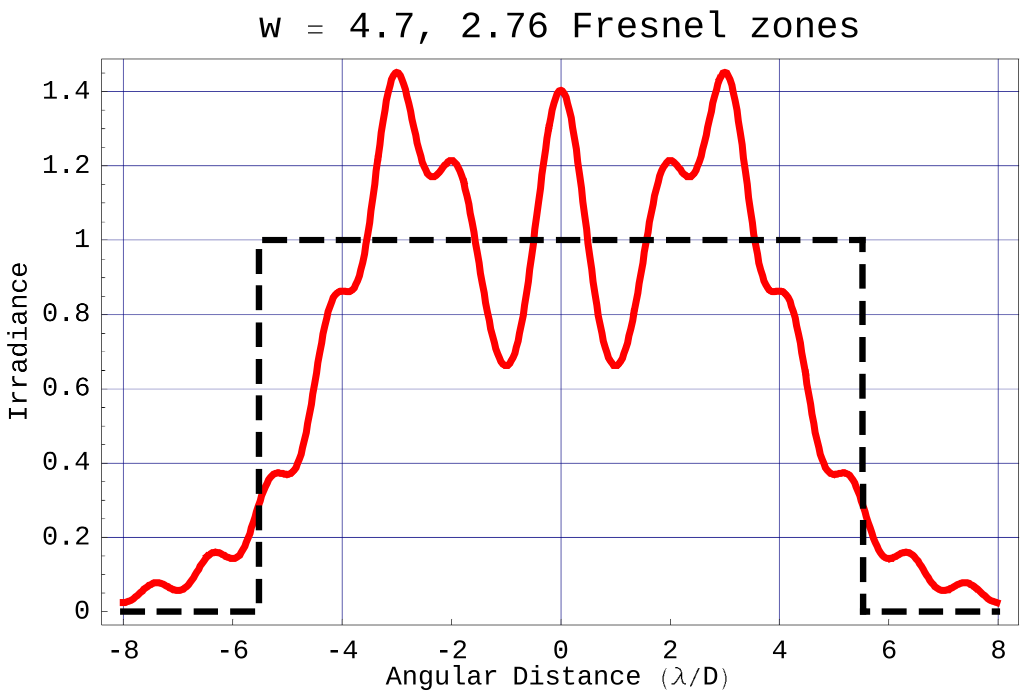
Plots of Fresnel diffraction of rectangular aperture

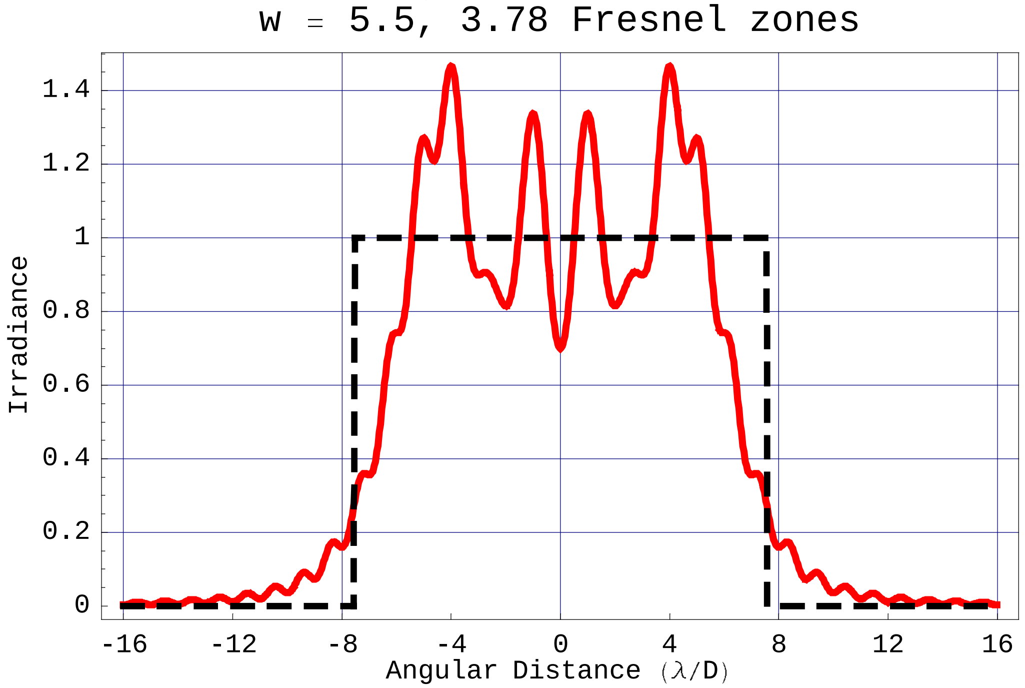

w = 0.6, 0.045 Fresnel zones

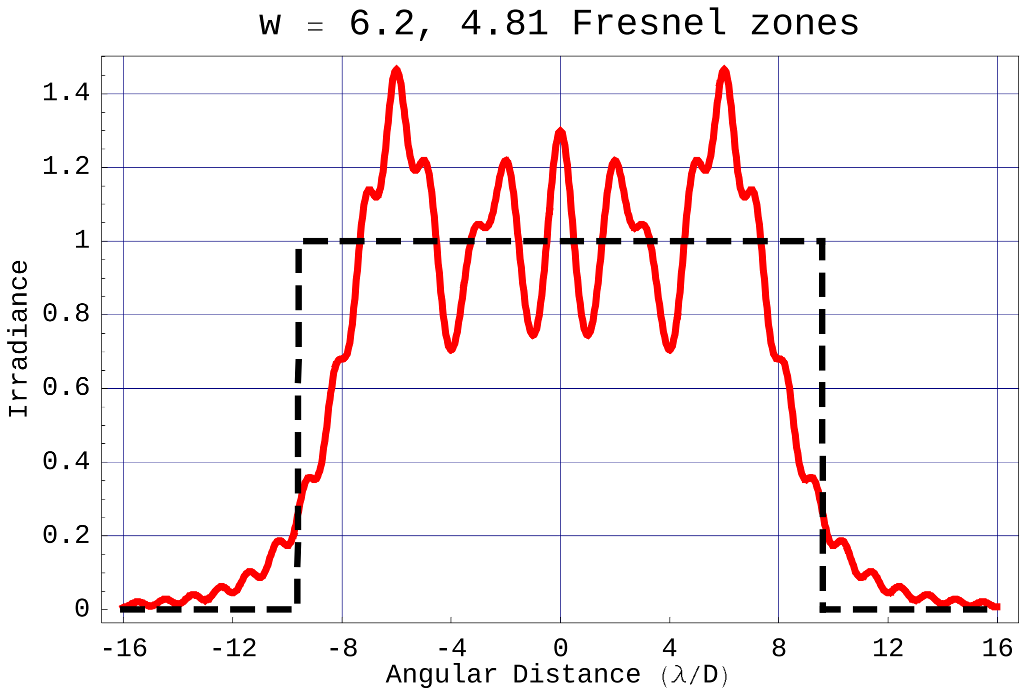

w = 1.5, 0.281 Fresnel zones

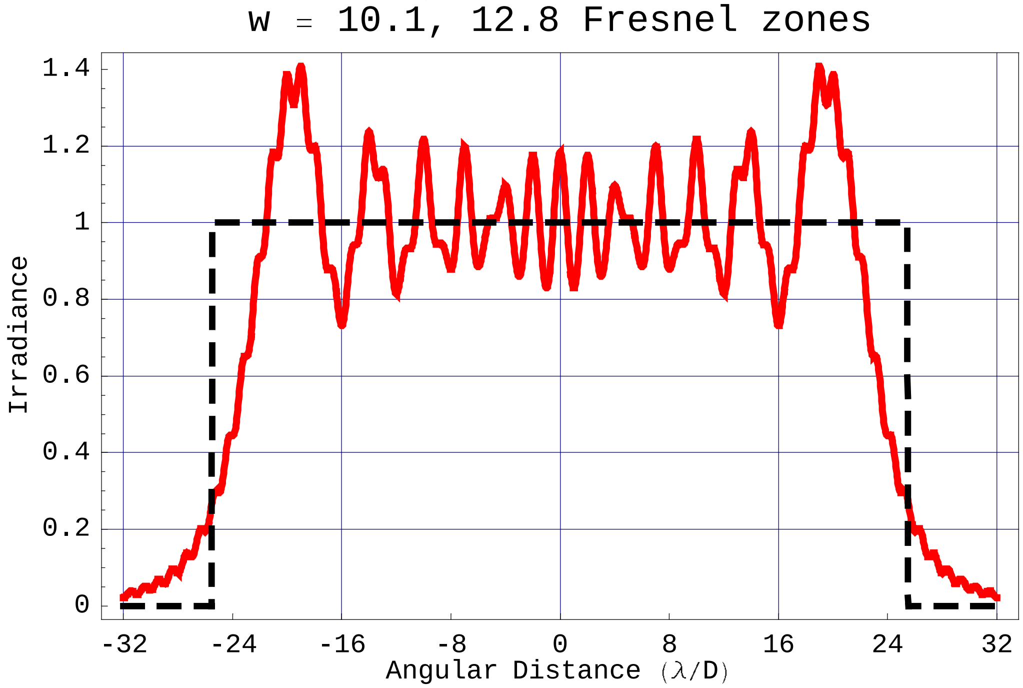

w = 2.4, 0.72 Fresnel zones


w = 3.7, 1.71 Fresnel zones


w = 4.7, 2.76 Fresnel zones
Irradiance
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-8
-6
-4
-2
0
2
4
6
8
Angular Distance (λ/D)


w = 5.5, 3.78 Fresnel zones
Irradiance
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-16
-12
-8
-4
0
4
8
12
16
Angular Distance (λ/D)





Plot Options

Fourier Transform Modules